Wicksellian Theory of Forest Rotation under Interest Rate Variability

Luis H. R. Alvarez[1] and Erkki Koskela[2]

December 18, 2002

[1] Department of Economics, Quantitative Methods in Management, Turku School of Economics and Business Administration, FIN-20500 Turku, Finland, e-mail: luis.alvarez@tukkk.fi

[2] Department of Economics, University of Helsinki, FIN-00014 University of Helsinki, Finland, e-mail: erkki.koskela@helsinki.fi

Abstract

The current literature on optimal forest rotation makes the assumption of constant interest rate. However, the irreversible harvesting decisions of forest stands are typically subject to relatively long time horizons over which interest rate do fluctuate. In this paper we apply the Wicksellian single rotation framework to extend the existing studies to cover the unexplored case of variable and stochastic interest rate. Given the technical generality of the considered valuation problem, we provide a mathematical characterization of the two dimensional optimal stopping problem and develop several new results. We show that allowing for interest rate uncertainty increases the optimal rotation period when the value of the optimal policy is a convex function of the current interest rate, provide plausible conditions under which this holds, and establish that increased interest rate volatility lengthens the optimal rotation period. Finally, and importantly, allowing for interest rate uncertainty as a mean reverting process and forest value as a geometric Brownian motion we provide an explicit solution for the two dimensional path-dependent optimal stopping problem. Numerical illustrations indicate that interest rate volatility has a significant quantitative importance.

Keywords: Wicksellian rotation, stochastic interest rates, optimal stopping, free boundary problems.

JEL Subject Classification: Q23, G31, C61

AMS Subject Classification: 91B76, 49K15, 49J15

1 Introduction

In forest economics the well-known model by Faustmann 1849 has been the most often used starting point in studies considering the optimal rotation period of forest stands. Under the assumption of constant timber prices, constant total cost of clear-cutting and replanting as well as constant interest rate, perfect capital markets and perfect foresight the model leads to a constant rotation period for an even aged stand, which maximizes the present value of forest stand over an infinite time horizon (see e.g. Clark 1976, Johannsson and Löfgren 1985 and Samuelson 1976). The representative rotation age depends on timber price, total cost of clear-cutting and replanting, nature of forest growth as well as the interest rate.

The basic assumptions and predictions of the Faustmann model do not seem to lie in conformity with empirical evidence (see e.g. Kuuluvainen and Tahvonen 1999). This has led to ongoing research, which has extended the basic Faustmann model under perfect foresight to allow for amenity valuation of timber (see e.g. Hartman 1976), the potential interdependence of forest stands as producers of amenity services (see e.g. Koskela and Ollikainen, 2000, 2001) as well as imperfect capital markets (see e.g. Tahvonen and Salo and Kuuluvainen 2001). The resulting rotation age has been shown to depend on the properties of amenity valuation function, the nature of stand interdependencies and potential borrowing constraints in the capital markets. In particular, in the latter case all the basic properties of optimal forest harvesting become different than the ones in the classical Faustmann model.

Finally, the perfect foresight assumption has been relaxed in studies focusing on the implications of stochastic timber prices (see e.g. Brazee and Mendelsohn 1988, Thomson 1992, Plantinga 1998, and Insley 2002), risk of forest fire (see e.g. Reed 1984) and/or stochastic forest growth on optimal rotation age (see e.g. Reed 1993, Miller and Voltaire 1983, Morck and Schwartz and Stangeland 1989, Clarke and Reed 1989, 1990, Willassen 1998 and Alvarez 2001 b). The effect of uncertainties on the optimal rotation period depends on the type of uncertainty. In the case of forest fire risk modelled as a Poisson process the rotation age will become shorter due to the higher effective discount rate (see Reed 1984) while in the presence of timber price and/or forest growth risk usually the reverse happens; higher risk in price or in age-dependent growth will tend to lengthen the rotation period by lowering the effective discount rate (see e.g. Clarke and Reed 1989, Willassen 1998 and Alvarez 2001 b).

This rotation literature has covered several interesting cases and provided useful insights. There is, however, a very important issue, which has not yet been analyzed. To our knowledge in all the research associated with optimal rotation periods of forest stands the assumption of constant interest rate has been sticked to. As we know from empirical research, interest rates fluctuate over time and the implications of this empirical finding for the term structure of interest rates, asset pricing etc. have been one of the major research areas in financial economics (for an up-to-date theoretical and empirical survey in the field see Cochrane 2001, chapter 19; see also Björk 1998, chapter 17 for an extensive treatment of interest rate modelling). If the investment projects would be very liquid ones, then interest rate fluctuations would not necessarily matter very much. In the case of forestry, however, the situation is different. Given the relatively slow growth rate of forests, investing in replanting is a long-term investment project, over which the expected behavior of the interest rate as the opportunity cost will be important. Similarly, since many real investments are productive over a considerably long time period, we are tempted to argue that the variability of interest rates should play a key role in the rational valuation and exercise policies of real irreversible investment opportunities as well. Ingersoll and Ross 1992 have analyzed the effect of interest rate uncertainty on the timing of investment but they model the interest rate process as a martingale (i.e. as a process which has no drift). Alvarez and Koskela 2001 generalizes their findings by allowing for stochastic interest rate of a mean reverting type.

In this paper we analyze the unexplored issue of what is the impact of variable and stochastic interest rate on optimal forest rotation. Since our main emphasis is to consider the impact of a stochastic interest rate on the optimal rotation policy, we first model the underlying interest rate dynamics as a general one factor diffusion process without explicit parametrization of the model. In this way, we plan to establish robust results valid for most well-established one factor interest rate models appearing in the financial literature (cf. Björk 1998 chapter 17, Black and Karasinski 1991, Cox, Ingersoll, and Ross 1980, 1981, 1985, Ingersoll and Ross 1992, Merton 1973, 1975, and Vasiček 1977). We also model the stochastic interest rate as an explicitly parametrized mean reverting process and the forest value in a simpler form to provide an explicit solution. We show among others that allowing for interest rate uncertainty will increase the optimal rotation period under the natural condition when the value of the optimal policy is convex in terms of the current interest rate and provide plausible conditions under which this holds. We also establish that increased interest rate volatility will increase the value of the optimal policy and move the exercise date further, meaning that the rotation period becomes longer. Finally, modelling interest rate uncertainty as a mean reverting process and forest value as a geometric Brownian motion, we provide an explicit solution for the two dimensional path-dependent optimal stopping problem. Numerical illustrations indicate that interest rate volatility has a significant quantitative importance.

We proceed as follows: In section 2 we present a framework to study the Wicksellian single rotation problem in the thus far unexplored situation of stochastic interest rate variability. Since the problem is more general than the constant discounting case, we first provide a mathematical characterization of the optimal rotation policy and its value, and then state the main results. Section 3 provides a solvable model when we specify interest rate uncertainty as a mean reverting process and forest value as a geometric Brownian motion. Section 4 presents some concluding remarks.

2 The Wicksellian Rotation Problem under Interest Rate Uncertainty

In this section we formulate the Wicksellian rotation problem in more general terms that usually by allowing stochastic interest rate variability. We proceed as follows. First we provide a set of sufficient conditions under which the optimal rotation problem admits a unique solution and under which the value of optimal policy can be obtained from an associated boundary value problem subject to standard value matching and smooth fit (or smooth pasting) conditions. Second, we analyze the relationship between increased volatility and the optimal rotation period.

In what follows, we model the stochastic interest rate dynamics as a general one factor diffusion model without explicitly parametrizing the drift of the underlying dynamics. This is because our purpose is to explore the impact of interest rate uncertainty on optimal rotation under very general assumptions in order to be able to establish robust results which would be valid for most well-established one factor interest rate models appearing in the literature of financial economics (cf. Björk 1998 chapter 17, Black and Karasinski 1991, Cox, Ingersoll, and Ross 1980, 1981, 1985, Ingersoll and Ross 1992, Merton 1973, 1975, and Vasiček 1977). In line with these arguments, we assume that the interest rate process $\{r_t; t \geq 0\}$ is defined on a complete filtered probability space $(\Omega, P, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ satisfying the usual conditions and that r_t is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$dr_t = \alpha(r_t)dt + \sigma(r_t)dW_t, \quad r_0 = r, \tag{2.1}$$

where W_t denotes standard Brownian motion, $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}$ is continuously differentiable with a Lipschitz continuous derivative, and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping for guaranteeing the existence of a solution for (2.1) (at least continuous; cf. Borodin and Salminen 1996, pp. 46–47). In order to avoid interior singularities, we also assume that $\sigma(r) > 0$ for all $r \in (0, \infty)$, that ∞ is a natural boundary for the diffusion r_t (non-explosive paths), and that 0 is either unattainable or exit for r_t (cf. Borodin and Salminen 1996, pp. 14–19). It is worth observing that if both boundaries are unattainable and

$$\int_0^\infty m'(y)dy < \infty,$$

where $m'(r) = 2/(\sigma^2(r)S'(r))$ denotes the density of the speed measure m of the diffusion r_t and

$$S'(r) = \exp\left(-\int \frac{2\alpha(r)}{\sigma^2(r)}dr\right)$$

denotes the density of the scale function of the diffusion r_t, then r_t will tend towards a long run steady state distributed according to the stationary distribution with density (cf. Borodin and Salminen 1996, pp. 35–36, see also Merton 1975)

$$p(r) = \frac{m'(r)}{\int_0^\infty m'(y)dy}.$$

Having presented the dynamics describing the evolution of the interest rate, we now specify the dynamics for the forest value as follows

$$dX_t = \mu(X_t)dt, \quad X_0 = x \in \mathbb{R}_+, \tag{2.2}$$

where $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ is a known Lipschitz-continuous mapping measuring the growth rate of the forest value. It is now clear that given our assumptions on the underlying dynamics the differential operator associated with the two-dimensional process (X_t, r_t) now reads as

$$\mathcal{A}_\sigma = \frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} + \mu(x)\frac{\partial}{\partial x} + \alpha(r)\frac{\partial}{\partial r}.$$

Given the stochastic interest rate dynamics (2.1) and the deterministic forest value dynamics (2.2) we next consider the following Wicksellian stochastic single rotation problem (an optimal stopping problem)

$$V_\sigma(x,r) = \sup_\tau E_{(x,r)} \left[\int_0^\tau e^{-\int_0^s r_t dt} \pi(X_s) ds + e^{-\int_0^\tau r_s ds} g(X_\tau) \right], \tag{2.3}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time, $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuously differentiable and non-decreasing mapping denoting the payoff accrued from exercising the irreversible harvesting opportunity, and the mapping $\pi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, measuring the flow of returns accrued from leaving the harvesting opportunity unexercised, is assumed to be non-negative and continuous in terms of the forest value. The value function is denoted as $V_\sigma(x,r)$ in order to emphasize the relationship between volatility and the value of the optimal rotation policy. We can now restate the optimal rotation problem (2.3) by decomposing it into the immediate exercise payoff and the early exercise premium as is indicated by the observation

$$V_\sigma(x,r) = g(x) + F_\sigma(x,r),$$

where

$$F_\sigma(x,r) = \sup_\tau E_{(x,r)} \int_0^\tau e^{-\int_0^t r_s ds} \left[\pi(X_t) + \mu(X_t) g'(X_t) - r_t g(X_t) \right] dt \tag{2.4}$$

denotes the early exercise premium in the presence of interest rate uncertainty. Typically this kind of optimal stopping problems are solved by analyzing the variational inequalities characterizing the value of the optimal policy (cf. Brekke and Øksendal 1991, Øksendal and Reikvam 1998, and Øksendal 1998, pp. 214–215). In the present case we know from these studies that a function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ dominates the value function $V_\sigma(x,r)$ (i.e. $J(x,r) \geq V_\sigma(x,r)$) if, for example, $J(x,r)$ is continuously differentiable on $\mathbb{R}_+^2$, twice continuously differentiable outside a set $D \subset \mathbb{R}_+^2$ of measure zero, has locally bounded second order derivatives in a neighborhood of D and satisfies the variational inequalities

$$\begin{aligned} (\mathcal{A}_\sigma J)(x,r) - rJ(x,r) + \pi(x) &= 0, \qquad (x,r) \in \{(x,r) \in \mathbb{R}_+^2 : J(x,r) > g(x)\} \\ (\mathcal{A}_\sigma J)(x,r) - rJ(x,r) + \pi(x) &\leq 0, \qquad (x,r) \in \{(x,r) \in \mathbb{R}_+^2 : J(x,r) = g(x)\} \backslash D. \end{aligned}$$

Our main objective is to present a characterization of the comparative static properties of the optimal rotation policy and its value as functions of the volatility of the underlying interest rate process. To this end, we assume that the interest rate process $\{\hat{r}_t; t \geq 0\}$ is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$d\hat{r}_t = \alpha(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t)dW_t, \quad \hat{r}_0 = r, \tag{2.5}$$

where $\hat{\sigma} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping satisfying the inequality $\hat{\sigma}(r) \geq \sigma(r)$. Put somewhat differently, $\hat{r}_t$ can be interpreted as a diffusion evolving at the same rate as r_t but subject to greater stochastic fluctuations than r_t. We emphasize that although in most analyzes the comparison is between different versions (in terms of volatility) of a given underlying interest rate process, we also consider the cases where these processes may be different. In line with our previous notation, we denote as $V_{\hat{\sigma}}(x, r)$ the value of the optimal rotation policy and as $F_{\hat{\sigma}}(x, r)$ the early exercise premium in the presence of the more volatile interest rate dynamics $\hat{r}_t$. Our first result characterizing the sign of the relationship between increased volatility and the optimal rotation policy and its value is now summarized in

Theorem 2.1. *Assume that the value $V_\sigma(x, r)$ is convex as function of the current interest rate r. Then, $V_{\hat{\sigma}}(x, r) \geq V_\sigma(x, r)$ and $F_{\hat{\sigma}}(x, r) \geq F_\sigma(x, r)$ for all $(x, r) \in \mathbb{R}^2_+$. Moreover, $\{(x, r) \in \mathbb{R}^2_+ : V_\sigma(x, r) > g(x)\} \subset \{(x, r) \in \mathbb{R}^2_+ : V_{\hat{\sigma}}(x, r) > g(x)\}$. That is, if the value $V_\sigma(x, r)$ is convex as a function of the current interest rate r, then increased volatility increases both the value and the early exercise premium of the irreversible policy and, therefore, prolongs the optimal rotation period.*

Proof. See Appendix A. □

An economic interpretation of Theorem 2.1 goes as follows. Increased interest rate volatility means that the opportunity cost of not harvesting (i.e. leaving the harvesting opportunity unexercised) becomes more uncertain which will move the exercise date further into the future when the value $V_\sigma(x, r)$ of the optimal policy is a convex function of the current interest rate. That is, while increased volatility increases the expected present value of the future revenues it simultaneously increases the value of holding the opportunity alive. Since the latter effect dominates the former, the net effect of increased volatility is to lengthen the rotation period (cf. Dixit and Pindyck 1994, chapter 5).

Given the observation of Theorem 2.1, it is important to ask: under what conditions the value $V_\sigma(x, r)$ of the optimal policy under interest rate uncertainty is a convex function of the current interest rate. Before establishing our main characterization of the sign of the relationship between volatility and the optimal rotation policy, we present the following result characterizing both the convexity of the expected revenues and their dependence on the volatility of the underlying interest rate process.

Lemma 2.2. *Assume that $\sigma(r)$ is continuously differentiable with Lipschitz-continuous derivative, that the standard Novikov-condition*

$$E_r\left[e^{\frac{1}{2}\int_0^t \sigma'^2(r_s)ds}\right] < \infty \quad (t, r) \in \mathbb{R}^2_+$$

is satisfied, and that $\alpha(r)$ is concave. Then, the expected value

$$G_\sigma(t, x, r) = E_{(x,r)}\left[\int_0^t e^{-\int_0^s r_t dt}\pi(X_s)ds + e^{-\int_0^t r_s ds}g(X_t)\right]$$

is a decreasing and convex function of the current interest rate. Moreover, increased volatility of the underlying interest rate process increases its value.

Proof. See Appendix B. □

We next provide a set of sufficient conditions under which we can fix the sign of the relationship between the optimal rotation period and interest rate volatility. Our characterization is presented in

Theorem 2.3. *Assume that $\sigma(r)$ is continuously differentiable with Lipschitz-continuous derivative, that the standard Novikov-condition*

$$E_r\left[e^{\frac{1}{2}\int_0^t \sigma'^2(r_s)ds}\right] < \infty \quad (t,r) \in \mathbb{R}_+^2$$

is satisfied, that $\pi(x) \equiv 0$, and that $\alpha(r)$ is concave. Then, $V_{\hat{\sigma}}(x,r) \geq V_\sigma(x,r)$ and $F_{\hat{\sigma}}(x,r) \geq F_\sigma(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$, and $\{(x,r) \in \mathbb{R}_+^2 : V_\sigma(x,r) > g(x)\} \subset \{(x,r) \in \mathbb{R}_+^2 : V_{\hat{\sigma}}(x,r) > g(x)\}$. That is, increased volatility increases both the value and the early exercise premium of the irreversible policy and, therefore, prolongs the optimal rotation period in the absence of amenity valuation.

Proof. See Appendix C. □

According to Theorem 2.3 under quite plausible assumptions that the diffusion term is sufficiently smooth as a function of the interest rate and the drift term is a concave function of the interest rate, increasing interest rate volatility will lengthen the optimal rotation period in the absence of amenity valuation. Unfortunately, it is very difficult, if possible at all, to establish simple conditions under which the value of the optimal harvesting policy in the presence of amenity valuation would be a convex function of the current interest rate. More precisely, reconsider the valuation problem (2.3). An application of the strong Markov property of diffusions then shows that (provided that the functionals exist)

$$V_\sigma(x,r) = (R\pi)(x,r) + \sup_\tau E_{(x,r)}\left[e^{-\int_0^\tau r_s ds}(g(X_\tau) - (R\pi)(X_\tau, r_\tau))\right],$$

where

$$(R\pi)(x,r) = E_{(x,r)}\int_0^\infty e^{-\int_0^s r_t dt}\pi(X_s)ds$$

denotes the expected cumulative present value of the flow of revenues accrued from the amenity services. As is clear from this expression, in the presence of amenity valuation there are three components depending on the current interest rate r, not just one as in the absence of amenity services. Consequently, it is difficult to present simple conditions leading to an unambiguously negative relationship between uncertainty and rotation.

3 A Solvable Model for the Wicksellian Rotation Problem under Stochastic Interest Rate and Forest Value

In this section we provide an explicit solution for the two-dimensional path-dependent optimal stopping problem and illustrate our findings also numerically. More specifically, we model the stochastic interest rate dynamics as an explicitly parametrized mean reverting process (which lies in conformity with empirical evidence, see e.g. Cochrane 2001, chapter 19) and forest value in a simpler way by abstracting from amenity valuation.

Consider the following (path-dependent) optimal rotation problem

$$V(x,r) = \sup_{\tau} \mathbf{E}_{(x,r)} \left[e^{-\int_0^\tau r_s ds} X_\tau \right], \tag{3.1}$$

where the underlying processes (X_t, r_t) evolve according to the dynamics described by the following stochastic differential equations

$$dr_t = \alpha r_t(1-\gamma r_t)dt + \sigma r_t dW_t, \quad r_0 = r \tag{3.2}$$

and

$$dX_t = \mu X_t dt + \beta X_t d\hat{W}_t, \quad X_0 = x, \tag{3.3}$$

where $\alpha, \beta, \sigma, \gamma, \mu \in \mathbb{R}_+$ are known exogenously given constants and W_t and $\hat{W}_t$ are potentially correlated Wiener processes (under the objective probability measure $\mathbb{P}$) with a known correlation coefficient $\rho \in [-1, 1]$.

Having characterized the underlying stochastic dynamics and the considered Wicksellian optimal rotation problem, we are now in position to state the following.

Lemma 3.1. *The Wicksellian path-dependent single rotation problem (3.1) can be re-expressed in the path-independent form*

$$V(x,r) = xr^{-\frac{1}{\alpha\gamma}} \sup_{\tau} \mathbf{E}_r \left[e^{-\theta\tau} \hat{r}_\tau^{\frac{1}{\alpha\gamma}} \right], \tag{3.4}$$

where

$$\theta = \frac{1}{\gamma} - \mu - \frac{\sigma^2}{2\alpha\gamma}\left(1 + \frac{1}{\alpha\gamma}\right) + \frac{\sigma\beta\rho}{\alpha\gamma}$$

can be interpreted as a "risk-adjusted" discount rate and

$$d\tilde{r}_t = \left(\alpha + \beta\sigma\rho - \frac{\sigma^2}{\alpha\gamma} - \alpha\gamma\tilde{r}_t\right)\tilde{r}_t dt + \sigma\tilde{r}_t dW_t, \quad \tilde{r}_0 = r. \tag{3.5}$$

Proof. See Appendix D. □

In Lemma 3.1 we demonstrate how the original path-dependent single rotation problem can be transformed into an ordinary path-independent optimal stopping problem of a linear diffusion. Our main result in this section is now summarized in the following

Theorem 3.2. *Assume that the risk-adjusted discount rate is positive (i.e. $\theta > 0$). Then the value of the single rotation problem (3.1) reads as*

$$V(x,r) = xr^{-\frac{1}{\alpha\gamma}}\psi(r) \sup_{y \geq r} \left[\frac{y^{\frac{1}{\alpha\gamma}}}{\psi(y)} \right] = \begin{cases} x, & r \geq r^* \\ x\left(\frac{r^*}{r}\right)^{\frac{1}{\alpha\gamma}} \frac{\psi(r)}{\psi(r^*)}, & r < r^* \end{cases}$$

where

$$\psi(r) = r^\eta M\left(\eta, 2\eta + \frac{2a}{\sigma^2}, \frac{2\alpha\gamma}{\sigma^2} r\right),$$

$\eta = \frac{1}{2} - \frac{a}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{a}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}} > 0$, $a = \alpha + \beta\sigma\rho - \frac{\sigma^2}{\alpha\gamma}$, *and M denotes the standard confluent hypergeometric function (see e. g. [1]). Moreover, the optimal exercise threshold r^* is the unique root of the ordinary first order condition $\psi(r^*) = \alpha\gamma r^*\psi'(r^*)$. Especially, $r^* > \mu$ for all $\sigma > 0$ and $r^* = \mu$ when $\sigma = 0$.*

Proof. Since $L(r) = \sup_\tau \mathbf{E}_r \left[e^{-\theta\tau} \tilde{r}_\tau^{\frac{1}{\alpha\gamma}} \right]$ is an ordinary path-independent optimal stopping problem of a linear diffusion and, therefore, can be solved by relying on ordinary variational inequalities, the alleged result is a direct implication of Theorem 3 in Alvarez 2001 a. □

Theorem 3.2 demonstrates that the path-dependent optimal rotation problem (3.4) is explicitly solvable whenever the absence of speculative bubbles condition $\theta > 0$ is satisfied. It is worth noticing that in the absence of uncertainty the condition $\theta > 0$ can be simply expressed as $1/\gamma > \mu$ meaning that the steady-state interest rate exceeds the growth rate of forest value. On the other hand, under uncertainty about the interest rate and forest value the absence of speculative bubbles condition $\theta > 0$ can also be re-expressed as

$$\frac{1}{\gamma} > \mu + \frac{\sigma^2}{2\alpha\gamma}\left(1 + \frac{1}{\alpha\gamma}\right) - \frac{\sigma\beta\rho}{\alpha\gamma}.$$

Thus, we find that the condition $\theta > 0$ is strengthened by the presence of uncertainty whenever the correlation ρ between the two driving Brownian motions is non-positive and is weakened whenever the correlation is positive. Moreover, and importantly, higher volatility increases the required exercise premium and thus prolongs the optimal rotation period.

Remark: It is worth noticing that since

$$dX_t^b = \left(b\mu + \frac{1}{2}\beta^2 b(b-1)\right) X_t^b dt + b\beta X_t^b d\hat{W}_t,$$

the result of Theorem 3.2 can be applied for solving the associated optimal stopping problem

$$H(x,r) = \sup_\tau \mathbf{E}_{(x,r)} \left[e^{-\int_0^\tau r_s ds} X_\tau^b \right], \tag{3.6}$$

where $b \in \mathbb{R}$ is a known parameter measuring the curvature of the mapping x^b. As is clear from Theorem 3.2, in that case we find that provided that the absence of speculative bubbles condition $\tilde{\theta} = \frac{1}{\gamma} - b\mu - \frac{1}{2}\beta^2 b(b-1) - \frac{\sigma^2}{2\alpha\gamma}\left(1 + \frac{1}{\alpha\gamma}\right) + \frac{\sigma b\beta\rho}{\alpha\gamma} > 0$ is satisfied the value of the stopping problem (3.6) reads as

$$H(x,r) = x^b r^{-\frac{1}{\alpha\gamma}} \tilde{\psi}(r) \sup_{y \geq r} \left[\frac{y^{\frac{1}{\alpha\gamma}}}{\tilde{\psi}(y)} \right] = \begin{cases} x^b, & r \geq \tilde{r} \\ x^b \left(\frac{\tilde{r}}{r}\right)^{\frac{1}{\alpha\gamma}} \frac{\tilde{\psi}(r)}{\tilde{\psi}(\tilde{r})}, & r < \tilde{r} \end{cases}$$

where

$$\tilde{\psi}(r) = r^{\tilde{\eta}} M\left(\tilde{\eta}, 2\tilde{\eta} + \frac{2\tilde{a}}{\sigma^2}, \frac{2\alpha\gamma}{\sigma^2} r\right),$$

$\tilde{\eta} = \frac{1}{2} - \frac{\tilde{a}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\tilde{a}}{\sigma^2}\right)^2 + \frac{2\tilde{\theta}}{\sigma^2}} > 0$, and $\tilde{a} = \alpha + b\beta\sigma\rho - \frac{\sigma^2}{\alpha\gamma}$. Moreover, the optimal exercise threshold $\tilde{r}$ is the unique root of the ordinary first order condition $\tilde{\psi}(\tilde{r}) = \alpha\gamma\tilde{r}\tilde{\psi}'(\tilde{r})$.

Finally, we characterize the quantitative significance of the volatility coefficient σ by numerical illustrations. Assume that $\gamma = 25$, $\alpha = 0.07$, $\rho = 0$ and $\mu = 0.01$ (implying that for $\theta > 0$ the upper bound under which the absence of speculative bubbles condition is satisfied is $\sigma^* = 0.2585$). Then, the optimal threshold r^* and required exercise premium $r^* - \mu$ as a function of the underlying volatility coefficient are

σ	0.1	0.2	0.25	0.258
r^*	1.1%	1.58%	2.77%	4.37%
$r^* - \mu$	0.1%	0.58%	1.77%	3.37%

Table 1

According to the findings presented in Table 1 the required exercise premium increases from 0.1% to 3.37% as volatility increases from 0.1 to 0.258. In order to illustrate our results in the negative correlation case, we assume that $\gamma = 25$, $\alpha = 0.07$, $\rho = -0.5$ and $\mu = 0.01$ (implying that now $\sigma^* = 0.2286$). Then the optimal threshold and required exercise premium as a function of the underlying volatility coefficient are

σ	0.1	0.2	0.22
r^*	1.1%	1.86%	2.62%
$r^* - \mu$	0.1%	0.86%	1.62%

Table 2

Thus, we find that the required exercise premium increases from 0.1% to 1.62% as volatility increases from 0.1 to 0.22. According to these numerical illustrations, higher interest rate volatility has a very big effect on the required exercise premium, thus implying a significantly longer optimal rotation period.

4 Conclusions

There is currently an extensive literature about the determination of optimal forest rotation under various circumstances when amenity valuation of forest stands matters, when capital markets are imperfect so that landowners might be subject to credit rationing or when there is uncertainty about timber prices and/or forest growth due either to forest growth uncertainty or to risk of forest fire. Undoubtedly this literature has provided useful insights about the potential determinants of forest rotation. There is, however, an important issue, which has not yet been analyzed. To our knowledge all the literature makes a simplifying but in the forestry case an unrealistic assumption that the interest rate is constant. Clearly the irreversible harvesting decision of forest stands is a decision subject to a relatively long time horizon. Hence, given the relatively slow growth rate of forests, thinking about harvesting and investing in replanting is a long-term investment project over which the behavior of interest rates as the opportunity cost should matter a lot.

In this paper we have used the Wicksellian single rotation framework to extend the existing studies to cover the unexplored case of variable and stochastic interest rate. Since the problem is more general than the constant discounting case, we first provided a characterization of the optimal rotation policy as a two-dimensional path-dependent optimal stopping problem.

From an economic point of view we have established several new findings. First, we have demonstrated that allowing for interest rate uncertainty will increase the optimal rotation period under the condition that the value of the optimal policy is convex in terms of interest rate in the absence of uncertainty. Second, under the plausible assumptions that the diffusion term in the (Itô-) stochastic differential equation for the interest rate is sufficiently smooth as a function of the interest rate and the drift term is concave function of the interest rate, higher interest rate volatility will increase the value of waiting and prolong the optimal rotation period in the absence of amenity valuation. Third, modelling interest rate uncertainty as a mean reverting process and forest value as a geometric Brownian motion, we have provided an explicit solution for the two dimensional path dependent optimal stopping problem. Numerical illustrations indicate that interest rate volatility has a significant quantitative importance on the optimal rotation policy.

Whether our conclusions remain valid in the Faustmann's ongoing rotation problem is an open question beyond the scope of the present study. However, given the close connection of impulse control problems and optimal stopping theory (impulse control problems can be viewed as sequential stopping problems; cf. Alvarez 2001 b), we are tempted to conjecture that most probably our conclusions would remain valid with only minor modifications in the ongoing rotation case as well at least for some class of amenity valuation functions. Of course, the verification of this claim is still an open and challenging problem left for future research.

Acknowledgements: The research of Luis H. R. Alvarez has been supported by the *Foundation for the Promotion of the Actuarial Profession* and the *Yrjö Jahnsson Foundation*. Erkki Koskela thanks the *Research Unit of Economic Structures and Growth (RUESG)* in the University of Helsinki for financial support. The authors are grateful to two *anonymous referees* for helpful comments and to *Heikki Ruskeepää* for his assistance in the *MATHEMATICA*©-calculations. An earlier version of the paper was presented in the 2nd World Congress of Environmental and Resource Economists, Monterey, California, June 24–27, 2002.

References

[1] Abramowitz, M. and Stegun, I. A., *eds.*, *Handbook of mathematical functions*, 1968, Dover Publications, New York.

[2] Alvarez, L. H. R. *Solving optimal stopping problems of linear diffusions by applying convolution approximations*, 2001 a, *Mathematical Methods of Operations Research*, **53**, 89–99.

[3] Alvarez, L. H. R. *Stochastic forest growth and Faustmann's formula*, 2001 b, Turku School of Economics and Business Administration, mimeo.

[4] Alvarez, L. H. R. *Reward functionals, salvage values, and optimal stopping*, 2001 c, *Mathematical Methods of Operations Research*, **54**, 315–337.

[5] Alvarez, L. H. R. *On the form and risk-sensitivity of zero coupon bonds for a class of interest rate models*, 2001 d, *Insurance Mathematics and Economics*, **28**, 83–90.

[6] Alvarez, L. H. R. and Koskela, E. *Irreversible investment under interest rate variability*, 2001, mimeo.

[7] Black, F. and Karasinski, P. *Bond and option pricing when short rates are lognormal*, 1991, *Financial Analysts Journal*, **47**, 52–59.

[8] Björk, T. *Arbitrage Theory In Continuous Time*, 1998, Oxford UP, Somerset.

[9] Borodin, A. and Salminen, P. *Handbook on Brownian motion - Facts and formulae*, 1996, Birkhauser, Basel.

[10] Brazee, R. and Meldelsohn, R. *Timber harvesting with fluctuating prices*, 1988, *Forest Science*, **34**, 359–372.

[11] Brekke, K. A. and Øksendal, B. *The high contact principle as a sufficiency condition for optimal stopping*, 1991, In Lund, D. & Øksendal, B. (editors), *Stochastic Models and Option Values*, Elsevier/North-Holland, 187 – 208.

[12] Clark, C. W. *Mathematical Bioeconomics: The optimal management of renewable resources*, 1976, Wiley, New York.

[13] Clarke, H. R. and Reed, W. J. *The tree-cutting problem in a stochastic environment*, 1989, *Journal of Economic Dynamics and Control*, **13**, 569–595.

[14] Clarke, H.R. and Reed, W.J *Harvest decisions and asset valuations for biological resources exhibiting size-dependent stochastic growth*, 1990, *International Economic Review*, **31**, 147–169.

[15] Cochrane, J.H. *Asset Pricing*, 2001, Princeton University Press.

[16] Cox, J. C., Ingersoll, J. E., and Ross, S. A. *An analysis of variable rate loan contracts*, 1980, *Journal of Finance*, **35**, 389–403.

[17] Cox, J. C., Ingersoll, J. E., and Ross, S. A. *A re-examination of traditional expectation hypotheses about the term structure of interest rates*, 1981, *Journal of Finance*, **36**, 769–799.

[18] Cox, J. C., Ingersoll, J. E., and Ross, S. A. *A theory of the term structure of interest rates*, 1985, *Econometrica*, **53**, 385–407.

[19] Dixit, A. K. and Pindyck, R. S. *Investment under Uncertainty*, 1994, Princeton UP, Princeton.

[20] Dynkin, E. *Markov processes, Volume II*, 1965, Springer-Verlag, Berlin.

[21] Faustmann, M. *Berechnung des Wertes welchen Waldboden sowie noch nicht haubare Holzbestände für die Waldwirtschaft besitzen*, 1849, *Allgemeine Forst- unde Jagd-Zeitung*, **15**, 441–455.

[22] Hartman, R. *The harvesting decision when a standing forest has value*, 1976, *Economic Inquiry*, **14**, 52–58.

[23] Ingersoll, J. E., Jr. and Ross, S. A. *Waiting to invest: Investment and uncertainty*, 1992, *Journal of Business*, **65**, 1 – 29.

[24] Insley, M. *A real options approach to the valuation of a forestry investment*, 2002, *Journal of Environmental Economics and Management*, **44**, 471–492.

[25] Johansson, P. O. and Löfgren, K. G. *The economics of forestry and natural resources*, 1985, Basil Blackwell, Oxford.

[26] Koskela, E. and Ollikainen, M. *Forest taxation and rotation age under private amenity valuation: new results*, 2001, *Journal of Environmental Economics and Management*, **42**, 374–384.

[27] Koskela, E. and Ollikainen, M. *Optimal private and public harvesting under spatial and temporal interdependence*, 2001, *Forest Science*, **47**, 484–496.

[28] Kuuluvainen, J. and Tahvonen, O. *Testing the forest rotation model: evidence from panel data*, 1999, *Forest Science*, **45**, 539–551.

[29] Merton, R. C. *A rational theory of option pricing*, 1973, *Bell Journal of Economics and Management Science*, **41**, 141–183.

[30] Merton, R. C. *An Asymptotic Theory of Growth Under Uncertainty*, 1975, *Review of Economic Studies*, **42**, 375–393.

[31] Miller, R. A. and Voltaire, K. *A stochastic analysis of the three paradigm*, 1983, *Journal of Economic Dynamics and Control*, **6**, 371–386.

[32] Morck, R. and Schwartz, E. and Stangeland, D. *The valuation of forestry resources under stochastic prices and inventories*, 1989, *Journal of Financial and Quantitative Analysis*, **24**,473–487.

[33] Øksendal, B. *Stochastic differential equations: An introduction with applications*, (Fifth Edition) 1998, Springer, Berlin.

[34] Øksendal, B. and Reikvam, K. *Viscocity solutions of optimal stopping problems*, 1998, *Stochastics and Stochastics Reports*, **62**, 285 – 301.

[35] Plantinga, A. J. *The optimal timber rotation: an option value approach*, 1998, *Forest Science*, **44**, 192–202.

[36] Protter, P. *Stochastic integration and differential equations*, 1990, Springer, New York.

[37] Reed, W. J. *The effects of the risk of fire on the optimal rotation of a forest*, 1984, *Journal of Environmental Economics and Management*, **11**, 180–190.

[38] Reed, W. J. *The decision to conserve or harvest old-growth forest*, 1993, *Ecological Economics*, **8**, 45–69.

[39] Samuelson, P. A. *Economics of forestry in an evolving society*, 1976, *Economic Inquiry*, **14**, 466–492.

[40] Tahvonen, O. and Salo, S. and Kuuluvainen, J. *Optimal forest rotation and land values under a borrowing constraint*, 2001, *Journal of Economic Dynamics and Control*, **25**, 1595–1627.

[41] Thomson, T. A. *Optimal forest rotation when stumpage prices follow a diffusion process*, 1992, *Land Economics*, **68**, 329–342.

[42] Vasiček, O. *An equilibrium characterization of the term structure*, 1977, *Journal of Financial Economics*, **5**, 177–188.

[43] Willassen, Y. *The stochastic rotation problem: A generalization of Faustmann's formula to stochastic forest growth*, 1998, *Journal of Economic Dynamics and Control*, **22**, 573–596.

A Proof of Theorem 2.1

Proof. If the value $V_\sigma(x,r)$ is convex as function of r, then for all $(x,c) \in C_\sigma = \{(x,r) \in \mathbb{R}_+^2 : V_\sigma(x,r) > g(x)\}$ we have

$$E_{(x,r)}\left[e^{-\int_0^{\tau_n} \hat{r}_s ds} V_\sigma(X_{\tau_n}, r_{\tau_n})\right] = V_\sigma(x,r) + E_{(x,r)} \int_0^{\tau_n} e^{-\int_0^t \hat{r}_s ds} \left[(\mathcal{A}_{\hat{\sigma}} V_\sigma)(X_t, \hat{r}_t) - \hat{r}_t V_\sigma(X_t, \hat{r}_t)\right] dt,$$

where $\tau_n = n \wedge \inf\{t \geq 0 : \|(x,r)\| \geq n\} \wedge \tau(C_\sigma)$ is a sequence of almost surely finite stopping times converging towards $\tau(C_\sigma) = \inf\{t \geq 0 : (X_t, r_t) \notin C_\sigma\}$. Since

$$(\mathcal{A}_{\hat{\sigma}} V_\sigma)(x,r) - r V_\sigma(x,r) = \frac{1}{2}\left(\hat{\sigma}^2(r) - \sigma^2(r)\right) \frac{\partial^2 V_\sigma}{\partial r^2}(x,r) \geq 0$$

whenever $(x,c) \in C_\sigma$, we find that

$$V_\sigma(x,r) \leq E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} V_\sigma(X_{\tau_n}, r_{\tau_n})\right].$$

Letting $n \to \infty$ and invoking continuity of the value function $V_\sigma(x,r)$ across the boundary of the continuation region C_σ (cf. Dynkin 1965, Theorem 12.4, p. 7) then yields that

$$V_\sigma(x,r) \leq E_{(x,r)}\left[e^{-\int_0^{\tau(C_\sigma)} \hat{r}_s ds} g(X_{\tau(C_\sigma)})\right] \leq V_{\hat{\sigma}}(x,r),$$

proving that $V_\sigma(x,r) \leq V_{\hat{\sigma}}(x,r)$ on the continuation region C_σ. However, since $V_\sigma(x,r) = g(x)$ on $\mathbb{R}_+^2 \backslash C_\sigma$ and $V_{\hat{\sigma}}(x,r) \geq g(x)$ for all $(x,r) \in \mathbb{R}_+^2$, we find that $V_{\hat{\sigma}}(x,r) \geq V_\sigma(x,r)$ and $F_{\hat{\sigma}}(x,r) \geq F_\sigma(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. Moreover, if $(x,r) \in C_\sigma$, then $V_\sigma(x,r) \geq V_\sigma(x,r) > g(x)$ proving that $(x,r) \in C_{\hat{\sigma}} = \{(x,r) \in \mathbb{R}_+^2 : V_{\hat{\sigma}}(x,r) > g(x)\}$ as well. □

B Proof of Lemma 2.2

Proof. We follow the proof of Theorem 2 in Alvarez 2001 d. Denote now as $r_t(i)$, $t \geq 0$, the solution of the stochastic differential equation (2.1) subject to the initial condition $r_0 = i \in \mathbb{R}_+$. Given our smoothness assumptions $r_t(i)$ can be expressed in the (Itô-) form

$$r_t(i) = i + \int_0^t \mu(r_s(i))ds + \int_0^t \sigma(r_s(i))dW_s. \tag{B.1}$$

Given our assumptions, $r_t(i)$ constitutes a continuously differentiable mapping of i (this is based on the flow nature of the solution of a stochastic differential equation; cf. Protter 1990, Theorem V. 38 and 39). Define now the process $\{Y_t; t \geq 0\}$ as $Y_t = \partial r_t(i)/\partial i$. It is then well-known that (cf. Protter 1990, Theorem V. 39)

$$Y_t = 1 + \int_0^t \mu'(r_s(i))Y_s ds + \int_0^t \sigma'(r_s(i))Y_s dW_s. \tag{B.2}$$

Applying Itô's theorem to the mapping $y \mapsto \ln y$ then implies that the solution of the stochastic differential equation (B.2) can be expressed as

$$Y_t = \frac{\partial r_t(i)}{\partial i} = \exp\left(\int_0^t \mu'(r_s(i))ds\right) Z_t(1), \tag{B.3}$$

where, given our assumptions, the process $\{Z_t(1); t \geq 0\}$ defined as

$$Z_t(1) = \exp\left(\int_0^t \sigma'(r_s(i))dW_s - \frac{1}{2}\int_0^t \sigma'^2(r_s(i))ds\right)$$

is a positive martingale starting at date 0 from 1 for any possible $i \in \mathbb{R}_+$. The strong uniqueness of a solution for the stochastic differential equation

$$dZ_t = \sigma'(r_t(i))Z_t dW_t \quad Z_0 = 1$$

then, in turn, implies that $Z_t(1)$ is not affected by i. The concavity of the drift $\mu(r)$ then implies that $\mu'(r)$ is non-increasing in r and, therefore, that $\mu'(r_s(\rho)) \leq \mu'(r_s(i))$ for all $\rho \geq i$ and $s \in [0, t]$. Consequently, we find that $\partial r_t(i)/\partial i$ is non-increasing in i, proving the alleged concavity of the solution $r_t(i)$ as a function of i. Since a decreasing and convex transformation of an increasing and concave mapping is decreasing and convex, we observe that the discount factor $e^{-\int_0^t r_s ds}$ is a decreasing and convex function of the current interest rate. Hence, the mapping

$$G_\sigma(t, x, r) = E_{(x,r)}\left[\int_0^t e^{-\int_0^s r_t dt}\pi(X_s)ds + e^{-\int_0^t r_s ds}g(X_t)\right]$$

is a decreasing and convex function of the current interest rate r as well. Proving that increased volatility increases the value of $G_\sigma(t, x, r)$ is then analogous with the proof of Theorem 2.1 after invoking the Feynman-Kač-formula (cf. Øksendal 1998, p. 135). □

C Proof of Theorem 2.3

Proof. As was established in Lemma 2.2, the discount factor $e^{-\int_0^t r_s ds}$ is decreasing and convex as a function of the current interest rate r. Given this observation, define now the increasing sequence $\{V_n(x, r)\}_{n\in\mathbb{N}}$ iteratively as

$$V_0(x, r) = g(x), \quad V_{n+1}(x, r) = \sup_{t\geq 0} E_{(x,r)}\left[e^{-\int_0^t r_s ds}V_n(X_t, r_t)\right].$$

It is now clear that $V_1(x, r)$ is convex as a function of r since the maximum of a convex function is convex. Consequently, all elements in the sequence $\{V_n(x, r)\}_{n\in\mathbb{N}}$ are convex as functions of r. Since $V_n(x, r) \uparrow V_\sigma(x, r)$ as $n \to \infty$ (cf. Øksendal 1998, p. 200) we find that for all $\lambda \in [0, 1]$ and $r, \rho \in \mathbb{R}_+$ we have that

$$\lambda V_\sigma(x, r) + (1-\lambda)V_\sigma(x, \rho) \geq \lambda V_n(x, r) + (1-\lambda)V_n(x, \rho) \geq V_n(x, \lambda r + (1-\lambda)\rho).$$

Letting $n \to \infty$ then implies that $\lambda V_\sigma(x, r) + (1-\lambda)V_\sigma(x, \rho) \geq V_\sigma(x, \lambda r + (1-\lambda)\rho)$ proving the convexity of $V_\sigma(x, r)$. The alleged results follow then from Theorem 2.1. □

D Proof of Lemma 3.1

Proof. Since

$$X_t = x\exp((\mu - \beta^2/2)t + \beta\hat{W}_t)$$

we find by applying Itô's theorem to the mapping $r \mapsto \ln r$ that

$$\ln(r_t/r) = \left(\alpha - \frac{1}{2}\sigma^2\right)t - \alpha\gamma\int_0^t r_s ds + \sigma W_t$$

which in turn implies

$$e^{-\int_0^t r_s ds} = \left(\frac{r_t}{r}\right)^{\frac{1}{\alpha\gamma}} e^{\frac{(\sigma^2-2\alpha)t}{2\alpha\gamma} - \frac{\sigma W_t}{\alpha\gamma}}.$$

Hence, we observe that the present value of the forest stand reads as

$$e^{-\int_0^t r_s ds}X_t = x\left(\frac{r_t}{r}\right)^{\frac{1}{\alpha\gamma}} e^{-\left(\frac{1}{\gamma}-\mu-\frac{\sigma^2}{2\alpha\gamma}\left(1+\frac{1}{\alpha\gamma}\right)+\frac{\sigma\beta\rho}{\alpha\gamma}\right)t}M_t,$$

where

$$M_t = e^{\beta\hat{W}_t - \frac{\sigma}{\alpha\gamma}W_t + \left(\frac{\sigma\beta\rho}{\alpha\gamma} - \frac{1}{2}\beta^2 - \frac{\sigma^2}{2\alpha^2\gamma^2}\right)t}$$

is a positive exponential $\mathcal{F}_t$-martingale. Consequently, we find that the path-dependent Wicksellian optimal rotation problem (3.1) can be re-expressed as an ordinary path-independent optimal stopping problem

$$V(x,r) = xr^{-\frac{1}{\alpha\gamma}}\sup_\tau \mathbf{E}_{(x,r)}\left[e^{-\theta\tau}r_\tau^{\frac{1}{\alpha\gamma}}M_\tau\right], \tag{D.1}$$

where

$$\theta = \frac{1}{\gamma} - \mu - \frac{\sigma^2}{2\alpha\gamma}\left(1 + \frac{1}{\alpha\gamma}\right) + \frac{\sigma\beta\rho}{\alpha\gamma}$$

can be interpreted as a "risk-adjusted" discount rate. Defining the equivalent measure $\mathbb{Q}$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t$ we can now re-express (D.1) as follows

$$V(x,r) = xr^{-\frac{1}{\alpha\gamma}}\sup_\tau \mathbf{E}^{\mathbb{Q}}_{(x,r)}\left[e^{-\theta\tau}r_\tau^{\frac{1}{\alpha\gamma}}\right]. \tag{D.2}$$

However, given the strong uniqueness of a solution for the stochastic differential equation

$$dr_t = \left(\alpha + \beta\sigma\rho - \frac{\sigma^2}{\alpha\gamma} - \alpha\gamma r_t\right)r_t dt + \sigma r_t d\tilde{W}_t, \quad r_0 = r$$

where $\tilde{W}_t$ is a standard Brownian motion under the equivalent measure $\mathbb{Q}$, we finally find that the rotation problem (3.1) can be rewritten in the path-independent form (3.4) defined under the objective measure $\mathbb{P}$. □